Module 4 - Limit as *x* Approaches *a*

Introduction

In this module you will use graphical, numerical and symbolic methods to investigate limits. You will begin by finding specific tolerances for a function, and then you will move to the general definition of the limit of a function. You will also use the TI-89 limit(command to examine limits.

Lesson Index:

After completing this module, you should be able to do the following:

- Use the Intersection and Trace features of the TI-89
- Find the tolerance for *x* that produces a given tolerance for *y* in a function
- Understand the definition of the limit of a function
- Use the TI-89 limit(command to evaluate limits
- Describe limits symbolically, graphically and numerically

Lesson 4.1: Tolerances

A function can be described as a black box with an input and a corresponding output. Each input value x enters the box and then undergoes a transformation that produces a corresponding output value y. For continuous functions you can ensure that the output values will be close to a particular value of y if the input values are close enough to the corresponding value of x. In this lesson you will quantify the concept of closeness for a specific function.

Before you begin this lesson, perform the "New Problem" command and clear all graphs and plots in the Y= Editor.

Investigating Tolerances of *y* = $\sqrt{3x-5}$

For *y* = $\sqrt{3x-5}$, *y* is close to 2 when *x* is close to 3.

The key question is

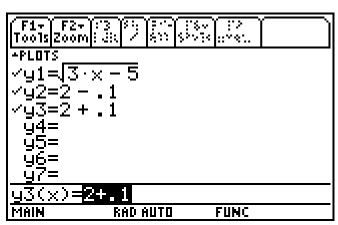
How close should *x* be to 3 to ensure that *y* is within 0.1 of 2?

Begin the investigation by graphing the function and the two horizontal lines that represent the *y*-tolerances: one line that is 0.1 below *y* = 2 and the other line that is 0.1 above *y* = 2.

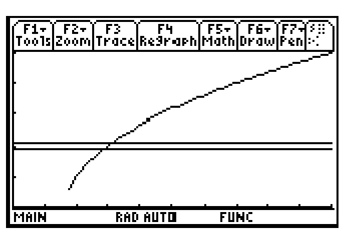
Graphing the Function and Output Bounds

In the Y= Editor enter

- The function: *y*1 = $\sqrt{3x-5}$
- The horizontal line 0.1 below 2: *y*2 = 2 – 0.1 = 1.9
- The horizontal line 0.1 above 2: *y*3 = 2 + 0.1 = 2.1



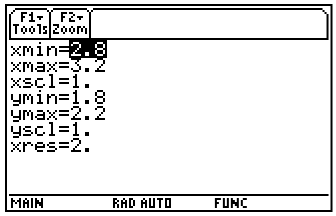
Display the graph of the function to see its basic shape and the graphs of the two horizontal lines that represent the output tolerance. Display the window with xmin = 0, xmax = 10, xscl = 1, ymin = 0, ymax = 5, and yscl = 1.



A Better View

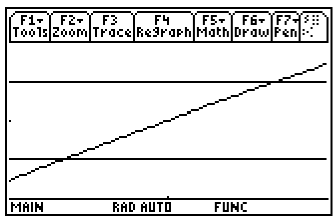
To better see the region of the graph where *y* is 0.1 unit away from 2, use viewing window values close to *x* = 3 and *y* = 2.

- Define the window values by entering the values shown below



An abbreviation for this window is [2.8, 3.2] x [1.8, 2.2].

- Display the graph by pressing [◆][GRAPH]

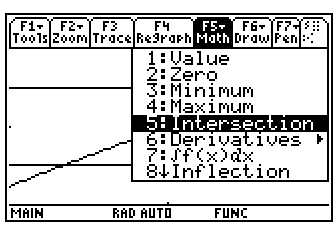


The graph of $y = \sqrt{3x-5}$ appears to be a diagonal line rather than a curve because a small portion of the graph has been magnified to fill this window. The points of intersection of the function and the lines have *y*-values that are 0.1 unit away from 2, and all points on the function between the two lines have *y*-values that are within 0.1 of 2.

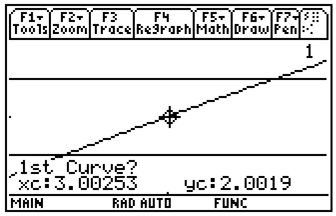
Finding the *x* Tolerances Using the Intersection Feature

The *x*-coordinates of the points where the graph of the function intersects the horizontal lines determine how close *x* needs to be to 3 so that *y* is within 0.1 of 2. The "Intersection" feature in the Math menu of the Graph screen can be used to find the intersection points.

- Display the Math menu by pressing [F5]



- Select 5:Intersection

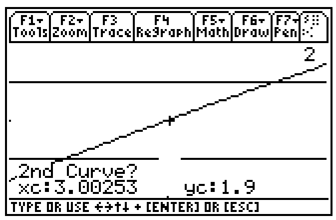


To find the x-value of the point of intersection of y2 = 1.9 (the lower bound of the *y*-tolerance) and y1= $\sqrt{3x-5}$(the graph of the function), you need to specify the two curves.

Selecting the Two Intersecting Curves

To find a point of intersection of two curves, the curves must be selected. The cursor is already on the function y1, as indicated by the small "1" in the upper right portion of the screen.

- Select the function as one of the curves by pressing ENTER



The small + shown on the graph of the function indicates that it is the first curve selected. The cursor is on the lower horizontal line to indicate that that line is currently chosen and it is blinking to indicate that it has not been selected. Note that "2" is shown in the upper right portion of the screen to indicate that the current position of the cursor is on the graph of y2.

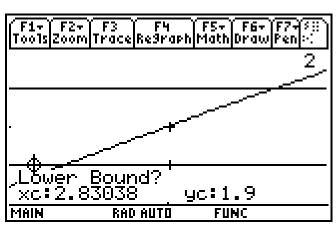
- Designate the lower line (y2) as the second curve by pressing ENTER

Specifying a Lower Bound

The TI-89 prompts you for a lower bound

- Choose a lower bound for the point of intersection by pressing and holding the ◀ key until the cursor is to the left of the *x*-coordinate of the point of intersection

  Any value less than the *x*-coordinate at the point of intersection will work.

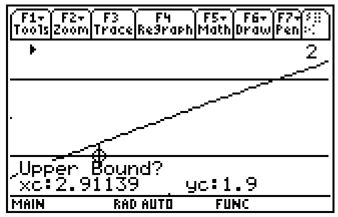


- Select the lower bound by pressing ENTER
  Notice the small right arrow at the lower bound.

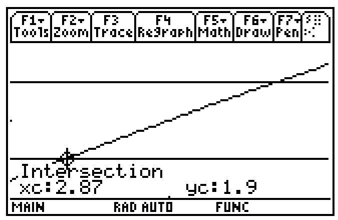
Specifying an Upper Bound

When prompted for an upper bound,

- Choose an upper bound for the intersection point by pressing and holding ▶ until the cursor is to the right of the *x*-coordinate of the intersection point



- Select that upper bound for the intersection point by pressing ENTER
  Any point to the right of the point of intersection will work.



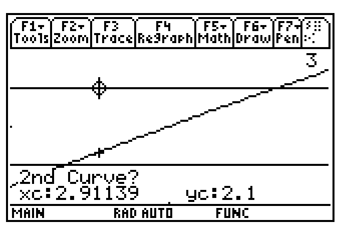
The *x*-coordinate of the point of intersection is 2.87.

Instead of using the arrow keys to select lower and upper bounds, one can just type in appropriate coordinates if they are known.

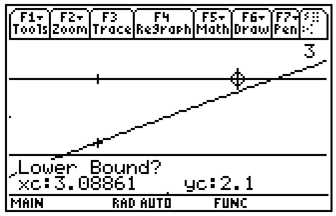
Finding the Second Point of Intersection

Now find the point where the graph of the function intersects the upper horizontal line, y3 = 2.1.

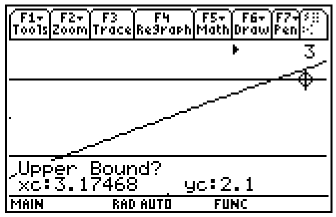
- Select Intersection from the Math menu by pressing F5 5
- Select the function as the first curve by pressing ENTER
- Move the cursor to the top line by pressing ▲



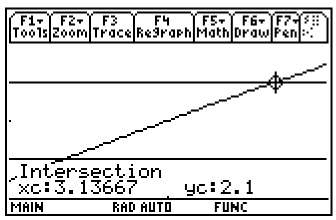
- Select y3 as the second curve by pressing ENTER
- Move the cursor until it is just to the left of the intersection point



- Select the lower bound by pressing ENTER
- Move the cursor until it is just to the right of the intersection



- Select the upper bound by pressing ENTER



The *x*-coordinate of this intersection point is approximately 3.13667.

You can conclude that if *x* is between 2.87 and 3.13666, then *y* will be within 0.1 of 2. But that was not the original question.

The original question was

> How close should *x* be to 3 to ensure that *y* is within 0.1 of 2?

It looks like there are two different answers. The left value of 2.87 is within 0.13 of 3. The right value of 3.13666 is within 0.13667 of 3.

4.1.1 Which value of *x*, 0.13 or 0.13666, will ensure that *y* is within 0.1 of 2?

Smaller Tolerances

4.1.2 For $f(x) = \sqrt{3x-5}$, how close should *x* be to 3 so that *y* is within 0.01 of 2?

Lesson 4.2: Definition of Limit

In the previous lesson you found tolerances graphically. In this lesson you will use the TI-89 computer algebra system to find these tolerances symbolically. This will prepare you to generalize the tolerances and develop the definition of limit.

Tolerances

The first tolerance you found in lesson 4.1 was in response to the question

How close should *x* be to 3 so that $\sqrt{3x-5}$ is within 0.1 of 2?

Restate the Question

Another way to phrase this question is

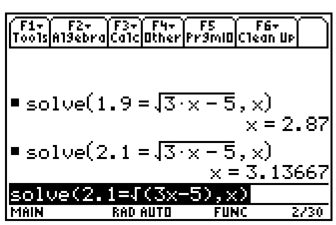
Find a positive number δ so that $1.9 < \sqrt{3x-5} < 2.1$ whenever $3 - \delta < x < 3 + \delta$.

Given *y*-Tolerance, Find *x*-Tolerance

The x-tolerance when the y-tolerance is 0.1 can be found by solving the

> *Definition*
>
> A compound (or extended) inequality is an inequality that compares more than two quantities and contains more than one inequality symbol.

compound inequality $1.9 < \sqrt{3x-5} < 2.1$. To find the values, use the solve(command to find the solutions to $1.9 = \sqrt{3x-5}$ and $\sqrt{3x-5} = 2.1$. Press ◆ ENTER or set the mode to AUTO.



The solution is approximately 2.87 < *x* < 3.13667.

Notice that the solutions are the same as those found in Lesson 4.1 when you used the intersection feature.

Finding δ

Compare $2.87 < x < 3.13667$ with the inequality $3 - \delta < x < 3 + \delta$. It must follow that $3 - \delta \approx 2.87$ and $3 + \delta \approx 3.13667$. However, solving these two equations for δ yields two different values: $\delta \approx 0.13$ and $\delta \approx 0.13667$. As in lesson 4.1, the smaller of the two values is the correct answer: $\delta \approx 0.13$.

Smaller Tolerance

The second tolerance you found in lesson 4.1 came from answering the question

How close should x be to 3 so that $\sqrt{3x-5}$ is within 0.01 of 2?

4.2.1 Rephrase this question with compound inequalities.

4.2.2 Use the solve(command to solve the first compound inequality.

4.2.3 Compare your answer to 4.2.2 with the inequality $3 - \delta < x < 3 + \delta$ and find a value of δ, the x-tolerance.

Finding a Generalized Solution

Suppose that you are asked to find values of δ that correspond to smaller and smaller tolerances for y around $y = 3$. Rather than going through the same process over and over for various y-tolerances, you could solve the problem once with a generalized y-tolerance. Call this general y-tolerance ε, the Greek letter "epsilon." The Greek letters delta, δ, and epsilon, ε, will be used to represent small *positive* numbers.

The tolerance question then becomes

How close should x be to 3 so that $\sqrt{3x-5}$ is within ε of 2?

Rephrasing the Question

The question can be rephrased using inequalities as

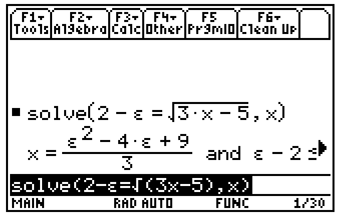
Find a positive number δ so that $2 - \varepsilon < \sqrt{3x-5} < 2 + \varepsilon$ whenever $3 - \delta < x < 3 + \delta$

Solving the Left Inequality: $2 - \varepsilon < \sqrt{3x-5}$

- Solve the left side of the inequality $2 - \varepsilon < \sqrt{3x-5} < 2 + \varepsilon$ with the command

  solve(2- ε= √(3x–5),x)

  The Greek letter ε can be entered in the TI-89 by pressing [♦] [(] [alpha] [÷].



The solution to the equation has two parts:

$x = \dfrac{\varepsilon^2 - 4\varepsilon + 9}{3}$ and $\varepsilon - 2 \le 0$.

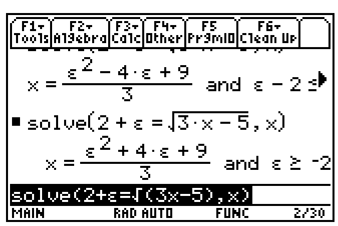
The second part of the solution, $\varepsilon - 2 \le 0$, is always true because ε is a *small positive* number.

> **Tech Tip!**
>
> **Entering Greek Letters**
>
> Greek letters are entered on the TI-89 by first pressing [♦] [(] and then entering the corresponding alpha key. For example, the Greek letter δ (Delta) is entered by pressing [♦] [(] [alpha] [,]. The alpha character above [,] is [D], which corresponds to δ.

Solving the Right Inequality: $\sqrt{3x - 5} < 2 + \varepsilon$

You can edit the command shown in the Edit Line to solve the right inequality by changing the "–" sign to "+."

- Edit the expression on the Edit Line to read solve(2+ ε= √(3x–5),x) by using the cursor movement keys and [←]
- Execute the command by pressing [ENTER]



The solution is $x = \dfrac{\varepsilon^2 + 4\varepsilon + 9}{3}$ and $\varepsilon \ge -2$.

Because ε is a small positive number, the statement $\varepsilon \ge -2$ is always true.

Combining the solutions to find δ

Combining the two solutions, in order to have 2 – ε < $\sqrt{3x-5}$ < 2 + ε we need

$$\frac{\varepsilon^2 - 4\varepsilon + 9}{3} < x < \frac{\varepsilon^2 + 4\varepsilon + 9}{3}$$

Comparing this inequality with 3 – δ < *x* < 3 + δ yields

$$3 - \delta = \frac{\varepsilon^2 - 4\varepsilon + 9}{3} \quad \text{and} \quad 3 + \delta = \frac{\varepsilon^2 + 4\varepsilon + 9}{3}$$

Solving these equations

for δ gives $\delta = -\frac{\varepsilon^2}{3} + \frac{4\varepsilon}{3}$ and $\delta = \frac{\varepsilon^2}{3} + \frac{4\varepsilon}{3}$

The smaller of these two values is $\delta = -\frac{\varepsilon^2}{3} + \frac{4\varepsilon}{3}$, which is the general solution. Given any value for ε you can use this equation to find the corresponding value for δ. For example, when ε = 0.01, this equation gives δ = 0.0133, the same value we found earlier.

Limits

When the values of the output can be made as close as we like to 2 by taking input values sufficiently close to 3, we say

The limit of the function *f(x)* = $\sqrt{3x-5}$ as *x* approaches 3 is equal to 2.

That is, the value of *f(x)* = $\sqrt{3x-5}$ gets closer and closer to 2 as *x* gets closer and closer to 3. The notation used to indicate this is

$$\lim_{x \to 3} \sqrt{3x-5} = 2$$

which is read "the limit of $\sqrt{3x-5}$ as x approaches 3 is 2."

In the previous example we found that $\delta = -\frac{\varepsilon^2}{3} + \frac{4\varepsilon}{3}$ guarantees that

$$2 - \varepsilon < \sqrt{3x-5} < 2 + \varepsilon \text{ whenever } 3 - \delta < x < 3 + \delta.$$

Because for any positive ε, a corresponding positive δ can be found that meets the conditions above,

$$\lim_{x \to 3} \sqrt{3x-5} = 2$$

Definition of Limit

Formally, $\lim_{x \to a} f(x) = L$ if for any $\varepsilon > 0$, however small, there exists a $\delta > 0$ such that

$$L - \varepsilon < f(x) < L + \varepsilon \text{ whenever } a - \delta < x < a + \delta$$

Conceptually, $f(x) \to L$ as $x \to a$.

4.2.4 Write the inequality with conditions that is associated with the limit. Interpret $\lim_{x \to 14} \sqrt{3x - 6} = 6$

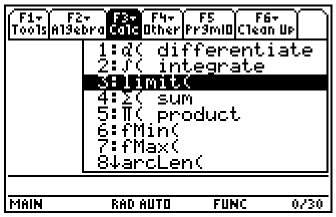
Lesson 4.3: TI-89 Limit Function

The TI-89 computer algebra system has a limit function. In this lesson you will use the TI-89 computer algebra system to find limits.

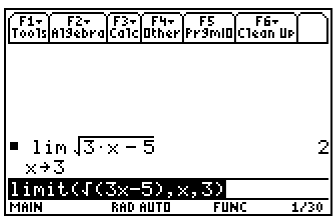
Using the Limit Function

From the Home screen, evaluate $\lim_{x \to 3} \sqrt{3x-5}$ by using the limit(command.

- Display the Calc menu by pressing [F3]



- Paste the limit(command to the Edit Line by pressing [3]
- Enter the function by pressing [2nd] [√] [3] [X] [−] [5] [)]
- Complete and execute the command by pressing [,] [X] [,] [3] [)] [ENTER]



So, $\lim_{x \to 3} \sqrt{3x-5} = 2$.

4.3.1 Use the limit(command to evaluate $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$. "Sin" is a yellow feature above the [Y] key and θ is a green feature above the [^] key. Be sure the calculator is in radian mode.

4.3.2 $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ is 1 in Radian mode. Find $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ in Degree mode.

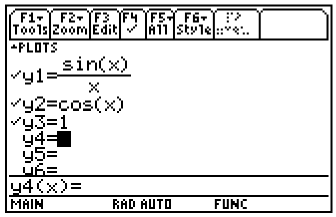
The Angle mode makes a difference in the evaluation of $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$. Most angle calculations in Calculus are done in Radian mode, so switch back to Radians now.

The Sandwich Theorem

The Sandwich Theorem is used in many Calculus books to prove $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. It can be shown that $\cos\theta < \frac{\sin\theta}{\theta} < 1$, and because the left and right terms in the inequality approach 1 as $\theta \to 0$, it must follow that the middle term also approaches 1.

This theorem can be visualized by graphing the three terms in the inequality simultaneously.

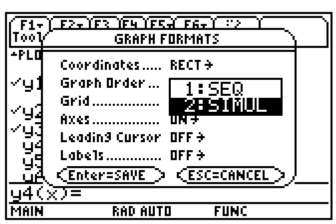
- Enter the three functions shown in the Y= menu below



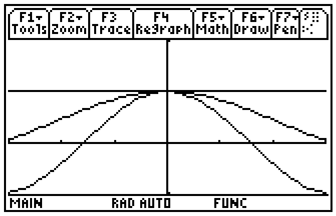
The graph format needs to be changed to simultaneous.

- From the Y= Editor display the Tools menu by pressing [F1]
- Highlight "Format" by using the cursor movement keys

- Open the Format submenu by pressing [ENTER]
- Display the Graph Order options by pressing (down)(right)
- Select "SIMUL" by pressing (down) and [ENTER]



- Save the Graph Format changes by pressing [ENTER]
- Display graphs of the functions in a [-3, 3] x [-1, 2] window



Seeing the three graphs simultaneously converge to 1 around x = 0 illustrates the Sandwich Theorem argument that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$ = 1. You can see the three functions converge again by pressing the Regraph button F4.

Left- and Right-Hand Limits

The limit(command can also be used to evaluate a left-hand limit or a right-hand limit, which is the value approached by a function as *x* approaches a specific value from the left or from the right.

4.3.3 Interpret $\lim_{x \to 0^-} \frac{1}{x}$ and evaluate it with the command limit(1/x,x,0,-1).

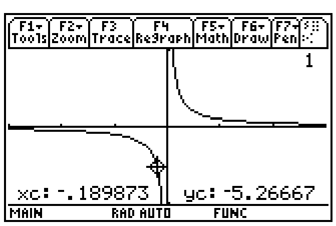
4.3.4 Interpret $\lim_{x \to 0^+} \frac{1}{x}$ and evaluate it by changing –1 to 1 in the Edit Line.

Using the Trace Feature

The Trace feature may be used to estimate limits. As the Trace cursor moves along a curve, the *x*- and *y*-values appear at the bottom of the Graph screen. Look at the *y*-values of *y* = $\frac{1}{x}$ as *x* approaches 0 from the left and from the right by using the Trace feature.

- Graph *y* = $\frac{1}{x}$ in a [-3, 3] x [-10, 10] window
- Activate the Trace cursor by pressing F3
- Display coordinates of the graph by repeatedly pressing ◀ or ▶ to move the cursor along the curve

Moving the Trace cursor toward zero from the left or the right provides dynamic graphical and numeric reinforcement for the left- and right-hand limits.



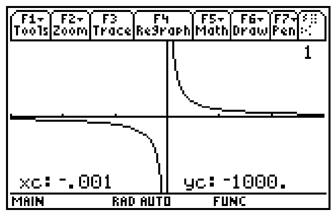
Entering a Specific *x*-Value

While the Trace cursor is active, you can move it to a particular point by typing in the *x*-coordinate of that point. For example, to move to the point with $x = -0.001$,

- Enter –0.001.

  This value should appear by "xc" in the lower left corner of the screen.

- Execute the command by pressing ENTER



The cursor is not visible since it is below the viewing window, but the coordinates of the cursor are shown giving further evidence that $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

Self Test

1. Find graphically: How close should x be to 1 so that $x^2 + 3$ is within 0.1 of 4?
2. Use inequalities to rephrase the question, "How close should x be to 1 so that $x^2 + 3$ is within 0.1 of 4?". Use the TI-89 solve(command to find the solution.
3. Use the TI-89 limit(command to evaluate the following limits.
    a. $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$
    b. $\lim_{\theta \to 0} \frac{\tan(3\theta)}{\theta}$ (radian mode)
    c. $\lim_{x \to 3^-} \frac{x + 1}{x - 3}$
    d. $\lim_{x \to 3^+} \frac{x + 1}{x - 3}$
4. Provide graphical and numerical support for $\lim_{x \to -2^+} \frac{x + 1}{x + 2}$ by graphing the function $y = \frac{x + 1}{x + 2}$ and using the Trace feature.