Circles, Polygons, and Star Polygons

Concepts
- Polar equations and graphs
- Parametric equations and graphs
- Properties of polygons and circles
- Number theory

Materials
- TI-83+ calculator
- Student activity sheet ”Circles, Polygons, and Star Polygons”

Introduction
When the TI-83+ is in the FUNC mode it can only graph functions. In fact, it is impossible to graph equations that do not represent functions using the Y= editor unless more than one equation are used.

This activity is in two parts. Part A, “Is it a Circle or a Polygon?”, investigates the importance of plotting enough points for an accurate representation of a curve. You will see that “selective sampling” can severely distort expected results. This strategy is used by statisticians to manipulate conclusions from a set of data.

Part B, “Star Polygons and Number Theory Investigations”, defines a star polygon and explains the meaning of the notation ,star (*n,s*). This part of the activity investigates patterns of various star polygons, compares them to regular polygons, and relates their characteristics to the Greatest Common Factor and cyclic patterns.

Student Activity Sheet

A. Is it a Circle or a Polygon?

When drawing a line or circle with pencil and paper, the pencil point is drawing an infinite number of points on a continuous curve. Drawings and graphs on a graphing calculator are constructed very differently. The calculator plots a series of points then connects the points with line segments.

1. Draw a polar circle with a radius of 8 units (R = 8).

a. Set the MODE to Pol and Degree (Fig. 1) – press [ENTER] after each change to the menu and [2nd] QUIT to return to the Home Screen

b. Change the FORMAT to PolarGC□ (Fig. 2)

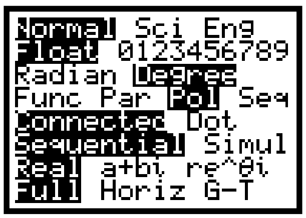


Figure 1

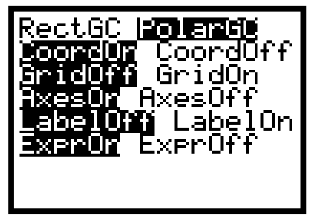


Figure 2

c. Change the WINDOW to match Figure 3. The θ-max indicates the total number of degrees to be used and θ-step indicates incremental steps in degrees that the calculator uses to plot the points. For θ-max = 360 one complete revolution is made in order to draw the polygon.

d. Enter the function r1 = 8 in the Y= editor (Fig. 4)

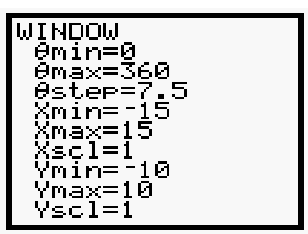


Figure 3

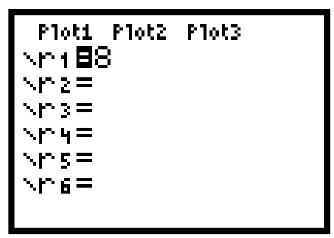


Figure 4

e. The equation draws a polar circle with a radius of 8 (R=8).

Although it looks like a circle it is actually a polygon with vertices plotted every 7.5 degrees (θ-step=7.5) around the origin and connected by line segments.

Since 360/7.5 = 48, the "circle" is really a polygon with 48 vertices (48-gon). (Fig 5)

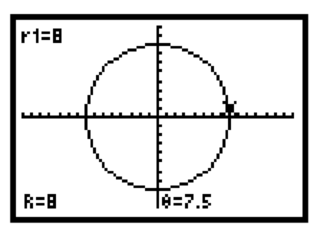


Figure 5

2. Draw another circle using the same settings as in #1 but change the θ-step = 1.
   a. How many vertices are used for the θ-step = 1? Write a formula to determine this.

   b. Draw a sketch of the polygon and compare it to the one in Figure 5.

   c. What effect does using a smaller θ-step have on the drawing speed of on the polygon?

3. Draw another circle using the same settings as in #1 but change the θ-step = 40.
   a. How many vertices does the new polygon have?

   b. Draw a sketch of the polygon and compare it to the one in Figure 5. Mark the vertices.

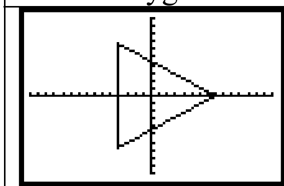
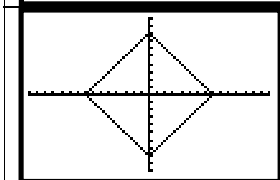
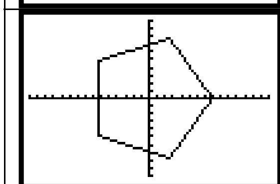
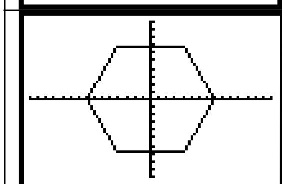
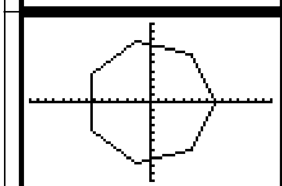
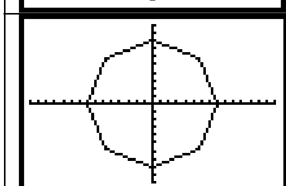
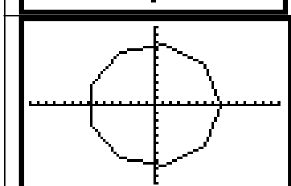
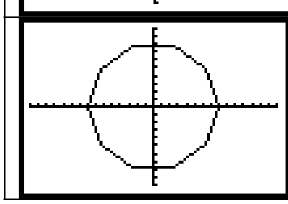
   c. What effect does using a larger θ-step have on the drawing speed of on the polygon?

4. What is the formula for determining the number of vertices based on the θ-step?

5. Write the formula for determining the θ-step based on the number of vertices.

6. Given the polygon and the number of degrees in one ‘orbit’ (θ-max), determine the θ-step needed to construct the polygons in Table 1.

Table 1

Polygon	Number of vertices	Polygon Name	Degrees in one ‘orbit’ (θ-max)	Incremental step (θ-step)
			360	
	4			
				72
		Hexagon		
				
				
				
				

7. Determine the measure of the interior angle of the polygons in Table 1 and give the corresponding value of θ-step. Fill in Table 2.

Table 2

Polygon	Triangle	Square	Pentagon	Hexagon	Heptagon	Octagon	Nonagon	Decagon
Interior Angle Measure								
θ-step								

8 What is the relationship between the θ-step and the measure of the interior angle?

9. Is it possible to set a θ-step so that the resulting figure is NOT a polygon? Explain

B. Star Polygons and Number Theory Investigations

All of the polygons in part A are simple polygons because the line segments do not cross over each other. A polygon for which this is not true is called a ***star polygon***. Figure 6 is an example of a star polygon.

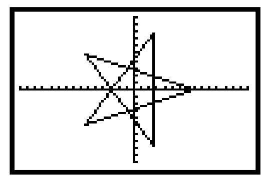


Figure 6

A star polygon, described by star(*n, s*), has *n* points on the circle and line segments that connect every s^{th} point (*s* must be less than *n*). With this notation, all of the simple polygons in Part A can be denoted as star (*n*, 1).

Unlike simple polygons that require only one 'orbit' (revolution) around the circle of points (vertices), a star polygon requires more than one 'orbit'. Another interpretation of *s* is the number of 'orbits'.

1. The star polygon in Figure 6 is a star (5, 2).
   a. How many points are there in star (5, 2)? ____________

      Make a conjecture about what the '*n*' represents in star (*n, s*)?

   b. Trace around the star polygon in Figure 7. Start at point A and continue to trace without lifting the pencil. Notice, for example, that one line segment connects points *A* and *C* and another connects points *E* and *B*.

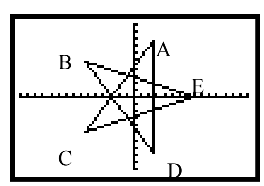


Figure 7

      Fill-in the blank:

      Line segments do not connect consecutive points. Instead, each line segment connects every _____________ point.

      Explain the relationship between your answer and the value *s* in star (*n, s*).

c. The simple polygons, star (n, 1), such as those in Part A need only one orbit to draw the entire figure. For these polygons θ-max = 360. Star (5, 2) requires more than one orbit. Begin tracing at point A and continue, without lifting your pencil, until you return to point A.

   How many 'orbits' are needed to trace the complete star (5, 2)? ______________

   Based on this, what θ-max is needed to draw a star (5, 2)? ____________________

   In general, what is θ-max for a star (n,s)? ______________________

d. Recall that θ-step determines the incremental step in degrees taken around the circle. In the pentagon, star (5, 1), there are 5 points with each point placed at a distance of 1/5 around the circle from the next point. Therefore, for star (5, 1), θ-step = 360/5 = 360 * 1/5 = 72.

   This holds true for any simple polygon of n vertices – each point is placed at a distance of 1/n from the next point. For star (n, 1,), θ-step = 360/n = 360 * 1/n.

   Fill-in the blanks: In star (5, 2), there are _________ vertices with each point placed at a distance of _____________ around the circle from the next point.

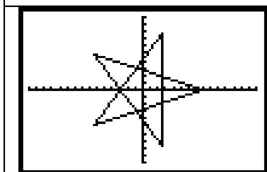
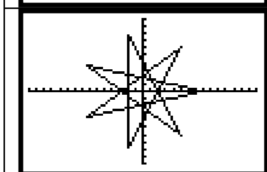
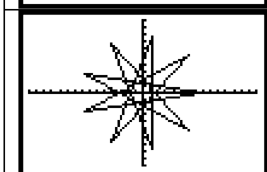
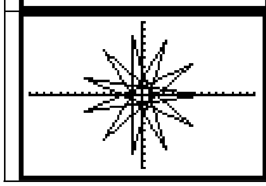
   From this information, star (5, 2) needs θ-step = _________________.

   In general, what is the θ-step for a star (n, s)? _____________________ .

e. Try it! Adjust the window settings for θ-max and θ-step and compare the graph with the star polygon in Figure 6. Draw your first attempt and then your last attempt.

2. By changing the θ-max and the θ-step you can construct other star polygons. Fill in Table 3 with the correct θ-max and θ-step needed to construct the indicated star polygons. Then verify by adjusting the window settings to draw each indicated star polygon.

Table 3

Star Polygon	Star (n, s)	Number of vertices (n)	Number of orbits (s)	Degrees in 'n' orbits (θ-max=360*s)	Incremental step (θ-step=360*s/n)
	Star (5, 2)	5	2	360*2=720	360*2/5=144
	Star (7, 3)				
	Star (9, 4)				
	Star (11, 5)				

Explain any difficulties you had in duplicating the graphs on your calculator.

3. GCF stands for "Greatest Common Factor".
   a. Fill-in Table 4 with the values of *n, s,* and GCF for the indicated star polygons.

Table 4

	Star (5, 2)	Star (7, 3)	Star (9, 4)	Star (11, 5)
n				
s				
GCF (*n, s*)				

b. Some unexpected results occur if *n* and *s* are not relatively prime (i.e., GCF (*n, s*) ≠ 1 Fill-in Table 5 and draw star (6, 2) and star (6, 3) in the left-most column. Label the points A – F.

Table 5

Star (n, s)	Number of vertices (n)	Number of orbits (s)	Degrees in 'n' orbits (θ-max=360*s)	Incremental step (θ-step=360*s/n)	Sketch
Star (6, 2)					
Star (6, 3)					

Explain how the graphs differ from those in Table 3.

4. A ‘family’ of star polygons has some interesting patterns.
   a. Construct each of the star polygons in Table 6. Draw a sketch of your graph and label the vertices A - G. Give the window dimensions you used to show the polygon.

Table 6

Star Polygon	(n)	(s)	θ-max	θ-step	Path of vertices	Sketch
star (7, 1)	7	1	360*1=360	360*1/7	A-B-C-D-E-F-G-A	
star (7, 2)	7	2	360*2=720	360*2/7	A-C-E-G-B-D-F-A	
star (7, 3)						
star (7, 4						
star (7, 5)						
star (7, 6)						

b. Which of the star polygons in Table 6 generate the same graph? What pattern do you see in the formulas for these polygons?

c. Compare the values *n* and *s*, the graphs, and paths of vertices for star (7, 3) and star (7, 4). Make a conjecture about the graphs of star (n, s) and star (n, r) where $r + s = n$. Use the information in Table 6 to verify your conjecture.

Teacher Notes

Introduction

The features and simplicity of graphing calculators brings the power of polar equations within reach of middle school and even elementary students.

In the traditional mathematics curriculum, polar equations are introduced to students during their second or third calculus course. Rarely is enough time allotted to develop a thorough understanding of the structure of these equations. Graphing is given little emphasis due to difficulties often encountered with hand-plotting. The advantages of using these graphing methods with younger students are two-fold. First, geometry and number theory lessons are enhanced with technology, and second, students will be better prepared for the more formal and rigorous lessons in their calculus classes.

This two-part activity is designed to apply geometric and algebraic thinking of polar graphing to draw polygons and star polygons. The first part, "Is it a Circle or a Polygon?", leads students to investigate the need for using a large number of data points when graphing an equation to avoid misleading results. This investigation illustrates the importance of plotting enough points for an accurate representation of a curve. As demonstrated, with "selective sampling", severely distorted results from what is expected can occur.

The second part, "Star Polygons and Number Theory Investigations", defines star polygons and demonstrates how the simple polygons in Part A are specialized cases of star polygons. Through investigations of characteristics of star polygons, applications of these characteristics to number theory such as Greatest Common Factor and cyclic patterns are made.

A viewscreen graphing calculator is needed. Both parts of the activity are appropriate for middle school through pre-calculus with slight changes made in terminology for the younger students. See **Additional Questions** for extensions to this activity.

Instructions

Part A: Is it a Circle or a Polygon?

1. Work with the students to draw the first circle. Students who have studied trigonometry can be reminded of what the notation in the window settings mean. For other students, these are explained in the narrative of this question.
2. a. There are 360 vertices. The formula is $\dfrac{360}{\theta\text{-step}} \rightarrow \dfrac{360}{1} \rightarrow 360 \text{ vertices}$.
   b. Similar to the one in Figure. 5.
   c. A smaller θ-step slows down the drawing but makes it more accurate.
3. a. There are 9 vertices.
   b. See Figure 8.

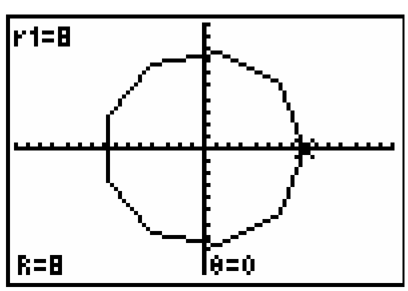


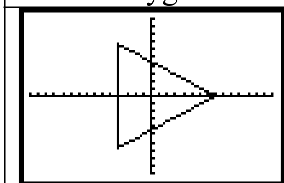
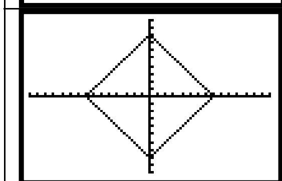
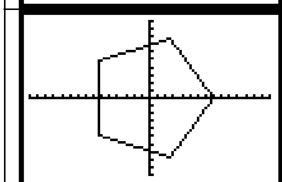
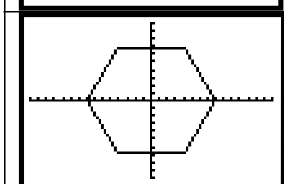
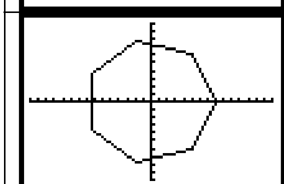
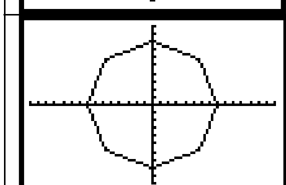
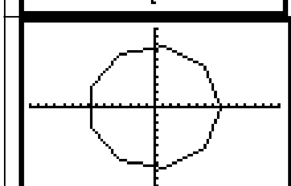
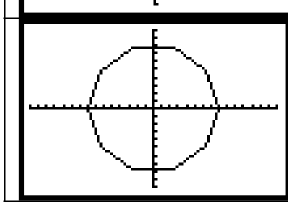
Figure 8

c. With a larger θ-step the drawing is faster but the drawing is not as accurate. It appears to be a nonagon (9-sided polygon).

4 -5 The formula is $\# \text{ vertices} = \dfrac{360}{\theta\text{-step}}$; the formula is $\theta\text{-step} = \dfrac{360}{\# \text{ vertices}}$

6. Have students show the computation of the Incremental step values in the table cell to assess understanding of the formula and approximate values.

Table 1.

Polygon	Number of vertices	Polygon Name	Degrees in one 'orbit' (θ-max)	Incremental step (θ-step)
	3	Triangle	360	$\frac{360}{3} = 120$
	4	Quadrangle	360	$\frac{360}{4} = 90$
	5	Pentagon	360	$\frac{360}{5} = 72$
	6	Hexagon	360	$\frac{360}{6} = 60$
	7	Heptagon	360	$\frac{360}{7} \approx 51.43$
	8	Octagon	360	$\frac{360}{8} = 45$
	9	Nonagon	360	$\frac{360}{9} = 40$
	10	Decagon	360	$\frac{360}{10} = 36$

7. Table 2

Polygon	Triangle	Square	Pentagon	Hexagon	Heptagon	Octagon	Nonagon	Decagon
Interior Angle Measure	60	90	108	120	128.57	135	140	144
θ-step	120	90	72	60	51.43	45	40	36

8. The θ-step is supplementary to the measure of the interior angle for a regular polygon.
9. Yes, if the θ-step is not a factor of θ-max (360, in this case), then the graph will not be closed; hence, not a polygon. For example, if θ-step = 150, the graph will be a segment that connects every 150^{th} vertex. To see this, turn off the axes in the FORMAT menu.

Part B: Star Polygons and Number Theory Investigations

1. a. There are 5 points in star (5, 2). The *'n'* in star (*n, s*) represents the number of vertices (or point).

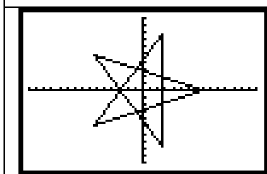
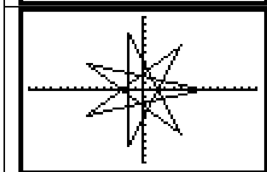
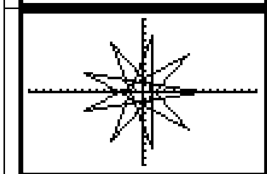
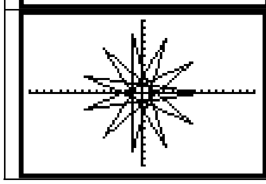
   b. Line segments do not connect consecutive points. Instead, each line segment connects every **second** vertex (or point).
   The *'s'* in star (*n, s*) indicates how many points (or vertices) that are skipped on each move.

   c. Two orbits are needed to trace the complete star (5, 2) with θ-max = 360*2 = 720. In general, for star (*n, s*), θ-max is the number of orbits needed to draw the complete figure; θ-max = 360 * *s*.

   d. In star (5, 2), there are **5** vertices with each point placed at a distance of **2/5** from the next point. From this information, star (5, 2) needs θ-step = **144**. This is computed as
   θ-step = $360 * \frac{2}{5} = 144$. In general, θ-step = $360 * \frac{s}{n}$.

   e. Answers will vary.

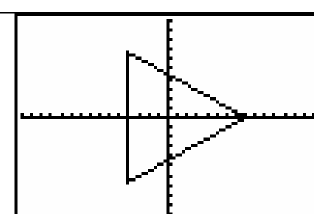
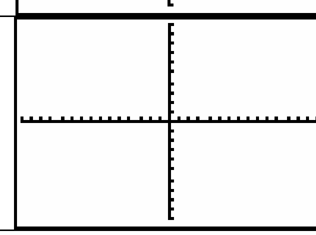
2. Table 3

Star Polygon	Star (n, s)	Number of vertices (n)	Number of orbits (s)	Degrees in 'n' orbits (θ-max = 360*s)	Incremental step (θ-step=360*s/n)
	Star (5, 2)	5	2	360*2 = 720	$360*\frac{2}{5}=144$
	Star (7, 3)	7	3	360*3 = 1080	$360*\frac{3}{7}\approx 154.286$
	Star (9, 4)	9	4	360*4 = 1440	$360*\frac{4}{9}=160$
	Star (11, 5)	11	5	360*5 = 1800	$360*\frac{5}{11}\approx 163.636$

3. a. Table 4

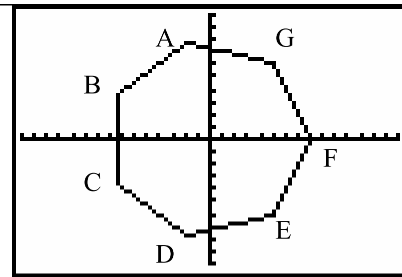
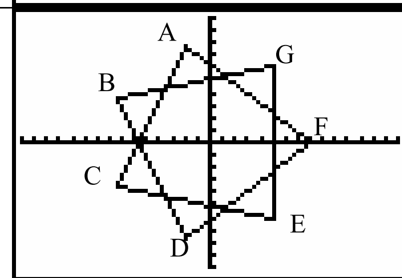
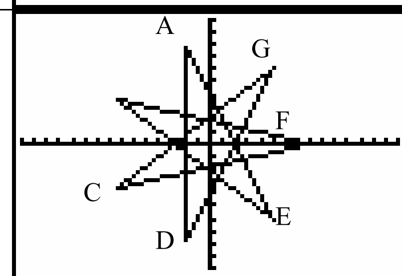
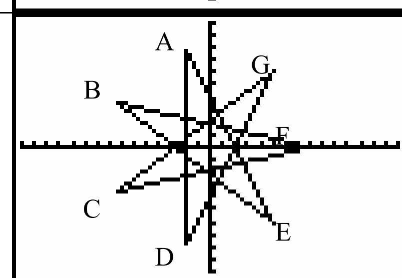
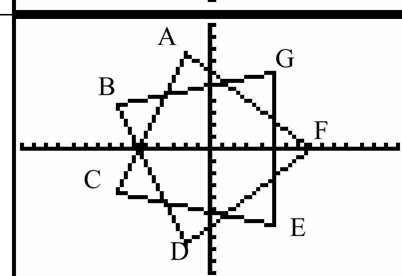
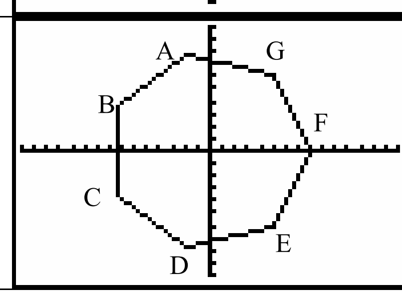
	Star (5, 2)	Star (7, 3)	Star (9, 4)	Star (11, 5)
n	5	7	9	11
s	2	3	4	5
GCF (*n, s*)	1	1	1	1

b. Table 5

Star (n, s)	Number of vertices (n)	Number of orbits (s)	Degrees in 'n' orbits (θ-max=360*s)	Incremental step (θ-step=360*s/n)	Sketch
Star (6, 2)	6	2	360*2 = 720	$360*\frac{2}{6}=120$	
Star (6, 3)	6	3	360*3 = 1080	$360*\frac{3}{6}=180$	

If GCF (n, s) = 1 then the figure will be a star polygon as illustrated in Table 3. If the GCF is not equal to 1, then the graph may replicate one of the star polygons in Table 3 or it may be a line segment. To see this, turn off the axes in the FORMAT menu.

4. Table 6

Star Polygon	(n)	(s)	θ-max	θ-step	Path of vertices	Sketch
star (7, 1)	7	1	360*1 = 360	360*1/7	A-B-C-D-E-F-G-A	
star (7, 2)	7	2	360*2 = 720	360*2/7	A-C-E-G-B-D-F-A	
star (7, 3)	7	3	360*3=1080	360*3/7	A-D-G-C-F-B-E-A	
star (7, 4	7	4	360*4=1440	360*4/7	A-E-B-F-C-G-D-A	
star (7, 5)	7	5	360*5=1800	360*5/7	A-E-B-F-C-G-D-A	
star (7, 6)	7	6	360*6=2160	360*6/7	A-F-D-B-G-E-C-A	

b. The following generate the same graph: star (7, 3) $\approx$ star (7, 4) , star (7, 2) $\approx$ star (7, 5) , star (7, 1) $\approx$ star (7, 6) . Star (n, s) and star (n, r) generate the same graphs when $s + r = n$.
c. The graphs are the same but their paths of vertices are in reverse order.

Additional Questions

This activity could be extended to the investigation of other polar graphs.

1. Determine the equation and window settings (use [-15, 15] by [/10, 10] for the x- and y-axes) to draw the graph in Fig. 9.

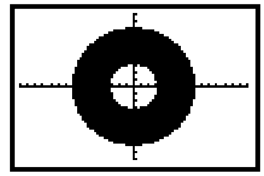


Figure 9

2. Continue exploring the relationship between θ-step and 360 in terms of GCF and LCM with the following problems.
   a. For what group of θ-steps will the graph be a polygon?
   b. For what group of θ-steps will the graph be a star polygon?
   c. For what group of θ-steps will the graph resemble a solid disk as in Fig. 8?

3. Repeat the explorations in Part A and B with the following polar equations. Begin with the window setting [-15, 15] by [/10, 10].
   a. **Roses**: Use **r = 8 sin (*n**θ)** and try different values for "*n*" until you can predict the number of petals. See Figure 10.
   Graph **r = 8 sin (3θ), r = 8 sin (4θ), r = 8 sin (5θ)**

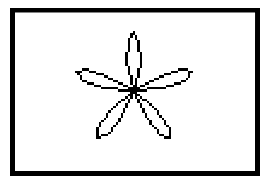


Figure 10

   b. **Archimedean Spirals**:
   Graph **r = 0.003θ**, set θ-max to 3000 then try different θ-steps. See Figure 11.
   Graph **r = 0.00003θ**, set θ-max to 30000 then try different θ-steps. See Figure 12.

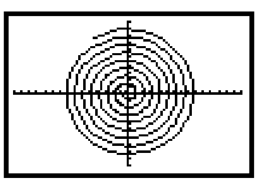


Figure 11

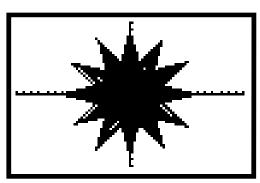


Figure 12

   c. **Logarithmic Spiral**:
   Graph $\mathbf{r = 2^{\theta/360}}$, set θ-max to 1280. Explain how a Logarithmic spiral is different from an Archimedean spiral.

d. Limacons: Use r = a + b*cos(θ). Explain what happens when a > b; a < b; and a = b. See Figure 13.
   Graph **r = 5 + 3cos(θ)**
   Graph **r = 5 + 5cos(θ)**
   Graph **r = 5 + 8cos(θ)**

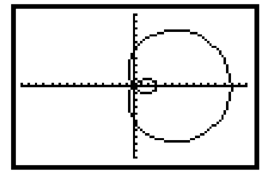


Figure 13

4. Many interesting curves can be graphed if we use parametric equations. In the MODE menu, select the Par(metric) mode.
   Lissajou Curves: Use **X = 8 * cos (a*T)** and **Y = 8 * sin (b * T)**
   Graph **X = 8 * cos (3*T), Y = 8 * sin (2 * T)**
   Graph **X = 8 * cos (2*T), Y = 8 * sin (3 * T)**
   Graph **X = 8 * cos (3*T), Y = 8 * sin (4 * T)**
   Graph **X = 8 * cos (7*T), Y = 8 * sin (5 * T)**

For further information about these concepts, read *Investigating Circles and Spirals with a Graphing Calculator* by Stuart Moskowitz, Mathematics Teacher, April 1994.